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WEAK AND STRONG CONVERGENCE THEOREMS FOR GENERALIZED HYBRID NONSELF-MAPPINGS IN HILBERT SPACES

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Abstract. In this paper, we first obtain fundamental results for a broad class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Then, we prove weak convergence theorems of Mann's type for the broad class of mappings in a Hilbert space. Furthermore, we prove two strong convergence theorems by hybrid methods for the class of the mappings in a Hilbert space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H and let T be a mapping of C into H. Then, we denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ is said to be *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in C$. A mapping $T : C \to H$ is called *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$\|Tx - y\| \leq \|x - y\|$$

for all $x \in C$ and $y \in F(T)$. A mapping $T : C \to H$ is called *nonspreading* [13] if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. Further, a mapping $T : C \to H$ is called *hybrid* [21] if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space. A mapping $F : C \to H$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [4] and Goebel and Kirk [6]. We also know that a firmly nonexpansive mapping F can be deduced from an equilibrium problem in a Hilbert space; see, for instance, [3] and [5]. From Baillon [2], and Takahashi and Yao [25], we know the following nonlinear ergodic theorem in a Hilbert space.

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Theorem 1.1. *Let H be a Hilbert space, let C be a nonempty closed convex subset of H and let T be a mapping of C into itself such that $F(T)$ is nonempty. Suppose that T satisfies one of the following:*

(i) *T is nonexpansive;*
(ii) *T is nonspreading;*
(iii) *T is hybrid;*
(iv) $2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$

Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$.

Aoyama, Iemoto, Kohsaka and Takahashi [1] introduced a class of mappings called λ-hybrid containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Very recently, Kocourek, Takahashi and Yao [11] introduced a more broad class of nonlinear mappings than the class of λ-hybrid mappings: A mapping $T : C \to H$ is called *generalized hybrid* if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Then, they proved a nonlinear ergodic theorem which generalizes cases of (i), (ii), (iii) and (iv), simultaneously. Further, they defined a class of nonlinear mappings called super hybrid containing the class of generalized hybrid mappings. We know that a super hybrid mapping is not quasi-nonexpansive generally.

In this paper, we deal with fundamental properties for generalized hybrid mappings and super hybrid mappings in a Hilbert space. Then, we prove weak convergence theorems of Mann's type [14] for super hybrid mappings in a Hilbert space. Further, we obtain strong convergence theorems for super hybrid mappings by using hybrid methods which were introduced by Nakajo and Takahashi [16], and Takahashi, Takeuchi and Kubota [23].

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a (real) Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. From [20], we know the following basic equality. For $x, y \in H$ and $\lambda \in \mathbb{R}$, we have

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2. \tag{2.1}$$

We also know that for $x, y, u, v \in H$,

$$2 \langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.2}$$

From (2.2), we have the following equality:

$$\|x - y + u - v\|^2 = \|x - y\|^2 + \|u - v\|^2 + 2 \langle x - y, u - v \rangle \tag{2.3}$$

$$= \|x-y\|^2 + \|u-v\|^2 + \|x-v\|^2 + \|y-u\|^2 - \|x-u\|^2 - \|y-v\|^2.$$

The following theorem is due to Opial [17].

Theorem 2.1. *Let H be a Hilbert space and let $\{x_n\}$ be a sequence of H such that $x_n \rightharpoonup x$. Then, for any $z \in H$ with $x \neq z$,*

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - z\|.$$

Let C be a nonempty closed convex subset of H. It is well-known that the set $F(T)$ of fixed points of a quasi-nonexpansive mapping $T : C \to H$ is closed and convex; see Ito and Takahashi [10]. Let C be a nonempty closed convex subset of H and $x \in H$. Then, we know that there exists a unique nearest point $z \in C$ such that $\|x - z\| = \inf_{y\in C} \|x - y\|$. We denote such a correspondence by $z = P_C x$. P_C is called the *metric projection* of H onto C. It is known that P_C is nonexpansive and

$$\langle x - P_C x, P_C x - u\rangle \geq 0$$

for all $x \in H$ and $u \in C$; see [20] for more details. Let C be a nonempty closed convex subset of H and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x,x) = 0, \quad \forall x \in C$;
(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0, \quad \forall x, y \in C$;
(A3) $\lim_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y), \quad \forall x, y, z \in C$;
(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [3] and [5].

Lemma 2.2. *Let C be a nonempty closed convex subset of H and let f be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists a unique $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Further, if

$$T_r x = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}, \quad \forall x \in H,\ r \in \mathbb{R},$$

then the following hold:

(1) *T_r is single-valued;*
(2) *T_r is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H.$$

Using (2) in Lemma 2.2 and (2.2), we have

$$\begin{aligned} 2\|T_r x - T_r y\|^2 &\leq 2\langle T_r x - T_r y, x - y\rangle \\ &= \|T_r x - y\|^2 + \|T_r y - x\|^2 - \|T_r x - x\|^2 - \|T_r y - y\|^2. \end{aligned}$$

So, for $y \in F(T_r)$ and $x \in H$, we have

$$\|T_r x - y\|^2 \leq \|y - x\|^2 - \|T_r x - x\|^2. \tag{2.4}$$

If $f(x,y)=0$, then we have $T_r = P_C$, i.e.,

$$\|P_C x - y\|^2 \le \|y-x\|^2 - \|P_C x - x\|^2 \tag{2.5}$$

for all $y \in C$ and $x \in H$, where P_C is the metric projection of H onto C.

For a sequence $\{C_n\}$ of nonempty closed convex subsets of a Hilbert space H, define s-Li$_n C_n$ and w-Ls$_n C_n$ as follows: $x \in$s-Li$_n C_n$ if and only if there exists $\{x_n\} \subset H$ such that $\{x_n\}$ converges strongly to x and that $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in$w-Ls$_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset H$ such that $\{y_i\}$ converges weakly to y and that $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies that

$$C_0 = \text{s-Li}_n C_n = \text{w-Ls}_n C_n, \tag{2.6}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [15] and we write $C_0 =$M-$\lim_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\cap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [15]. We know the following theorem [26].

Theorem 2.3. *Let H be a Hilbert space. Let $\{C_n\}$ be a sequence of nonempty closed convex subsets of H. If $C_0 =$M-$\lim_{n\to\infty} C_n$ exists and is nonempty, then for each $x \in H$, $\{P_{C_n} x\}$ converges strongly to $P_{C_0} x$, where P_{C_n} and P_{C_0} are the mertic projections of H onto C_n and C_0, respectively.*

3. Nonlinear operators

In this section, we first start with defining a wide class of nonlinear mappings containing the classes of nonexpansive mappings, nonspreading mappings, and hybrid mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty subset of H. Then, a mapping $T : C \to H$ is called *generalized hybrid* [11] if there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|Tx - y\|^2 + (1-\beta)\|x-y\|^2 \tag{3.1}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta)$-*generalized hybrid* mapping. Notice that the mapping above covers several well-known mappings. For example, an $(\alpha,\ \beta)$-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. We can also show that if $x = Tx$, then for any $y \in C$,

$$\alpha\|x - Ty\|^2 + (1-\alpha)\|x - Ty\|^2 \le \beta\|x - y\|^2 + (1-\beta)\|x-y\|^2$$

and hence $\|x - Ty\| \le \|x-y\|$. This means that an $(\alpha,\ \beta)$-generalized hybrid mapping with a fixed point is quasi-nonexpansive. Next, let us define a more general class of mappings than the class of generalized hybrid mappings in a Hilbert space. Let C be a nonempty subset of a Hilbert space H. A mapping $S : C \to H$ is called *super hybrid* [11] if there are $\alpha, \beta, \gamma \in \mathbb{R}$ such that

$$\begin{aligned} \alpha\|Sx - Sy\|^2 &+ (1-\alpha+\gamma)\|x - Sy\|^2 \\ &\le \big(\beta + (\beta-\alpha)\gamma\big)\|Sx - y\|^2 + \big(1 - \beta - (\beta-\alpha-1)\gamma\big)\|x-y\|^2 \\ &\qquad + (\alpha-\beta)\gamma\|x - Sx\|^2 + \gamma\|y - Sy\|^2 \end{aligned} \tag{3.2}$$

for all $x, y \in C$. We call such a mapping an $(\alpha,\ \beta,\ \gamma)$-*super hybrid* mapping. We notice that an $(\alpha,\ \beta,\ 0)$-super hybrid mapping is $(\alpha,\ \beta)$-generalized hybrid. So, the class of super hybrid mappings contains the class of generalized hybrid mappings. A super hybrid mapping is not quasi-nonexpansive generally. In fact, let us consider a super hybrid mapping S with $\alpha = 1$, $\beta = 0$ and $\gamma = 1$. Then, we have

$$\|Sx - Sy\|^2 + \|x - Sy\|^2 \leq -\|Sx - y\|^2 + 3\|x - y\|^2 + \|x - Sx\|^2 + \|y - Sy\|^2$$

for all $x, y \in C$. This is equivalent to

$$\|Sx - Sy\|^2 + 2\langle x - y, Sx - Sy\rangle \leq 3\|x - y\|^2$$

for all $x, y \in C$. In the case of $H = \mathbb{R}$, consider $Sx = 2 - 2x$ for all $x \in \mathbb{R}$. Then,

$$\begin{aligned}|Sx - Sy|^2 + 2\langle x-y, Sx - Sy\rangle &= |-2x + 2y|^2 + 2\langle x - y, -2x + 2y\rangle \\ &= 4|x - y|^2 + 4\langle x - y, y - x\rangle \\ &= 0 \leq 3|x - y|^2\end{aligned}$$

and hence S is super hybrid. However, S is not quasi-nonexpansive. Further, we have that

$$Tx = \frac{1}{2}Sx + \frac{1}{2}x = \frac{1}{2}(2 - 2x) + \frac{1}{2}x = 1 - \frac{1}{2}x$$

and hence T is nonexpansive. In general, we have the following theorem.

Theorem 3.1. *Let C be a nonempty subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \neq -1$. Let S and T be mappings of C into H such that $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, S is $(\alpha,\ \beta,\ \gamma)$-super hybrid if and only if T is $(\alpha,\ \beta)$-generalized hybrid. In this case, $F(S) = F(T)$.*

Proof. Put $\lambda = \frac{1}{1+\gamma} \neq 0$. Then, $T = \lambda S + (1 - \lambda)I$. We have that for any $x, y \in C$,

$$\begin{aligned}&\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2 \\ &\iff \alpha\|\lambda(Sx - Sy) + (1 - \lambda)(x - y)\|^2 + (1 - \alpha)\|\lambda(x - Sy) + (1 - \lambda)(x - y)\|^2 \\ &\qquad \leq \beta\|\lambda(Sx - y) + (1 - \lambda)(x - y)\|^2 + (1 - \beta)\|x - y\|^2.\end{aligned}$$

From (2.1), this inequalty is equivalent to

$$\begin{aligned}&\alpha(\lambda\|Sx - Sy\|^2 + (1 - \lambda)\|x - y\|^2 - \lambda(1 - \lambda)\|Sx - Sy - x + y\|^2) \\ &\qquad + (1 - \alpha)(\lambda\|x - Sy\|^2 + (1 - \lambda)\|x - y\|^2 - \lambda(1 - \lambda)\|y - Sy\|^2) \\ &\quad \leq \beta(\lambda\|Sx - y\|^2 + (1 - \lambda)\|x - y\|^2 - \lambda(1 - \lambda)\|x - Sx\|^2) + (1 - \beta)\|x - y\|^2 \\ &\iff \alpha(\lambda\|Sx - Sy\|^2 - \lambda\|x - y\|^2 - \lambda(1 - \lambda)\|Sx - Sy - x + y\|^2) \\ &\qquad + (1 - \alpha)(\lambda\|x - Sy\|^2 - \lambda\|x - y\|^2 - \lambda(1 - \lambda)\|y - Sy\|^2) \\ &\quad \leq \beta(\lambda\|Sx - y\|^2 - \lambda\|x - y\|^2 - \lambda(1 - \lambda)\|x - Sx\|^2) \\ &\iff \alpha\lambda\|Sx - Sy\|^2 + (1 - \alpha)\lambda\|x - Sy\|^2 \\ &\quad \leq \beta\lambda\|Sx - y\|^2 + (1 - \beta)\lambda\|x - y\|^2 - \beta\lambda(1 - \lambda)\|x - Sx\|^2 \\ &\qquad + (1 - \alpha)\lambda(1 - \lambda)\|y - Sy\|^2 + \alpha\lambda(1 - \lambda)\|Sx - Sy - x + y\|^2.\end{aligned}$$

Dividing by λ^2, we have from $\lambda^{-1} = \gamma + 1$ that

$$\begin{aligned}\alpha(\gamma+1)\|Sx-Sy\|^2 &+ (1-\alpha)(\gamma+1)\|x-Sy\|^2 \\ &\leq \beta(\gamma+1)\|Sx-y\|^2 + (\gamma+1)(1-\beta)\|x-y\|^2 - \beta\gamma\|x-Sx\|^2 \\ &\qquad + (1-\alpha)\gamma\|y-Sy\|^2 + \alpha\gamma\|Sx-Sy-x+y\|^2.\end{aligned}$$

We know from (2.3) that

$$\begin{aligned}\|Sx-Sy-x+y\|^2 = \|Sx-Sy\|^2 &- \|x-Sy\|^2 - \|Sx-y\|^2 \\ &+ \|x-y\|^2 + \|Sx-x\|^2 + \|Sy-y\|^2.\end{aligned}$$

So, we obtain

$$\begin{aligned}\alpha\|Sx-Sy\|^2 &+ \{(1-\alpha)+\gamma\}\|x-Sy\|^2 \\ &\leq \{\beta+(\beta-\alpha)\gamma\}\|Sx-y\|^2 + \{1-\beta-\gamma(\beta-\alpha-1)\}\|x-y\|^2 \\ &\qquad + (\alpha-\beta)\gamma\|x-Sx\|^2 + \gamma\|y-Sy\|^2.\end{aligned}$$

Then, S is (α, β, γ)-super hybrid if and only if T is (α, β)-generalized hybrid. From $T = \lambda S + (1-\lambda)I$, we also have $F(S) = F(T)$. This completes the proof. $\square$

From [11], we know the following fixed point theorem for generalized hybrid mappings in a Hilbert space.

Theorem 3.2. *Let C be a nonempty closed convex subset of a Hilbert space H and let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n z\}$ is bounded for some $z \in C$.*

As a direct consequence of Theorem 3.2, we have the following result.

Theorem 3.3. *Let C be nonempty bounded closed convex subset of a Hilbert space H and let T be a generalized hybrid mapping from C to itself. Then T has a fixed point.*

Using Theorems 3.1 and 3.3, we have the following fixed point theorem [11] for super hybrid mappings in a Hilbert space.

Theorem 3.4. *Let C be a nonempty bounded closed convex subset of a Hilbert space H and let α, β and γ be real numbers with $\gamma \geq 0$. Let $S : C \to C$ be an (α, β, γ)-super hybrid mapping. Then, S has a fixed point in C.*

4. Weak convergence theorem

In this section, we first prove a weak convergence theorem of Mann's type for super hybrid nonself-mappings in a Hilbert space. Before proving it, we need the following lemma for generalized hybrid nonself-mappings in a Hilbert space.

Lemma 4.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to H$ be a generalized hybrid mapping. Suppose that there exists $\{x_n\} \subset C$ such that $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Then, $z \in F(T)$.*

Proof. Since $T : C \to H$ is a generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2 + (1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2 + (1-\beta)\|x-y\|^2 \tag{4.1}$$

for all $x, y \in C$. Putting $T = I - A$, we have

$$\alpha\|x - Ax - (y - Ay)\|^2 + (1-\alpha)\|x - (y - Ay)\|^2 \\ \le \beta\|x - Ax - y\|^2 + (1-\beta)\|x - y\|^2$$

and hence

$$\alpha\{\|x-y\|^2 - 2\langle x-y, Ax - Ay\rangle + \|Ax - Ay\|^2\} \\ + (1-\alpha)\{\|x-y\|^2 + 2\langle x-y, Ay\rangle + \|Ay\|^2\} \\ \le \beta\{\|x-y\|^2 - 2\langle x-y, Ax\rangle + \|Ax\|^2\} + (1-\beta)\|x-y\|^2.$$

So, we have

$$\alpha\{-2\langle x-y, Ax-Ay\rangle + \|Ax\|^2 - 2\langle Ax, Ay\rangle + \|Ay\|^2\} \\ + (1-\alpha)\{2\langle x-y, Ay\rangle + \|Ay\|^2\} \\ \le \beta\{-2\langle x-y, Ax\rangle + \|Ax\|^2\}.$$

Then, we have

$$(\alpha-\beta)\|Ax\|^2 + \|Ay\|^2 \le 2\alpha\langle Ax, Ay\rangle + 2\langle x-y, (\alpha-\beta)Ax - Ay\rangle.$$

From $A = I - T$, we have

$$(\alpha-\beta)\|x - Tx\|^2 + \|y - Ty\|^2 \\ \le 2\alpha\langle x - Tx, y - Ty\rangle + 2(\alpha-\beta)\langle x-y, x-Tx\rangle - 2\langle x-y, y-Ty\rangle.$$

Suppose $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$. Let us consider

$$(\alpha-\beta)\|x_n - Tx_n\|^2 + \|z - Tz\|^2 \\ \le 2\alpha\langle x_n - Tx_n, z - Tz\rangle + 2(\alpha-\beta)\langle x_n - z, x_n - Tx_n\rangle \\ - 2\langle x_n - z, z - Tz\rangle.$$

Letting $n \to \infty$, we have $\|z - Tz\|^2 \le 0$. Then $Tz = z$. □

Using Lemma 4.1, we prove a weak convergence theorem of Mann's type [14] for super hybrid mappings in a Hilbert space. The proof is due to the technique developed by Ibaraki and Takahashi [7] and [8].

Theorem 4.2. *Let H be a Hilbert space, let C be a closed convex subset of H and let P_C be the metric projection of H onto C. Let α, β and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping with $F(S) \neq \emptyset$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \le \alpha_n \le 1$ and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$. Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\big(\alpha_n x_n + (1-\alpha_n)(\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n)\big), \quad n = 1, 2, \dots.$$

Then, the sequence $\{x_n\}$ converges weakly to an element v of $F(S)$, where $v = \lim_{n\to\infty} P_{F(S)}x_n$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Proof. Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have from Theorem 3.1 that T is an $(\alpha,\ \beta)$-generalized hybrid mapping and $F(S) = F(T)$. Let $z \in F(T)$. Since T is quasi-nonexpansive, we have

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_C(\alpha_n x_n + (1-\alpha_n)Tx_n) - z\|^2 \\ &\leq \|\alpha_n x_n + (1-\alpha_n)Tx_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. So, we have that $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}\|x_{n+1} - z\|^2 &= \|P_C(\alpha_n x_n + (1-\alpha_n)Tx_n) - z\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|Tx_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &\leq \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \\ &= \|x_n - z\|^2 - \alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2.\end{aligned}$$

So, we have

$$\alpha_n(1-\alpha_n)\|Tx_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Since $\lim_{n\to\infty} \|x_n - z\|^2$ exists and $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, we have $\|Tx_n - x_n\|^2 \to 0$. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$. By Lemma 4.1, we obtain $v \in F(T)$. Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know that $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|^2$ and $\lim_{n\to\infty} \|x_n - v_2\|^2$ exist. Put

$$a = \lim_{n\to\infty} (\|x_n - v_1\|^2 - \|x_n - v_2\|^2).$$

Note that for $n = 1, 2, \ldots,$

$$\|x_n - v_1\|^2 - \|x_n - v_2\|^2 = 2\langle x_n, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2.$$

From $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$, we have

$$a = \lim_{i\to\infty} (\|x_{n_i} - v_1\|^2 - \|x_{n_i} - v_2\|^2) = 2\langle v_1, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2 \tag{4.2}$$

and

$$a = \lim_{j\to\infty} (\|x_{n_j} - v_1\|^2 - \|x_{n_j} - v_2\|^2) = 2\langle v_2, v_2 - v_1\rangle + \|v_1\|^2 - \|v_2\|^2. \tag{4.3}$$

Combining (4.2) and (4.3), we obtain $0 = 2\langle v_2 - v_1, v_2 - v_1\rangle$ and hence $\|v_2 - v_1\|^2 = 0$. So, we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element v of $F(T)$. Since $\|x_{n+1} - z\| \leq \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Takahashi and Toyoda [24] that $\{P_{F(T)}x_n\}$ converges strongly to an element p of $F(T)$. On the other hand, we have from the property of $P_{F(T)}$ that

$$\langle x_n - P_{F(T)}x_n, P_{F(T)}x_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $P_{F(T)}x_n \to p$, we obtain

$$\langle v - p, p - u \rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} P_{F(T)}x_n$. This completes the proof. □

As direct consequences of Theorem 4.2, we obtain the following results.

Corollary 4.3. *Let H be a Hilbert space, let C be a closed convex subset of H and let P_C be the metric projection of H onto C. Let γ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be an (2, 1, γ)-super hybrid mapping, i.e.,*

$$2\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq \|x - Sy\|^2 + \|Sx - y\|^2 + 2\gamma\|x - y\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. *Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\big(\alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n)\big), \quad n = 1, 2, \dots.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(S)$, where $v = \lim_{n\to\infty} P_{F(S)}x_n$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Corollary 4.4. *Let H be a Hilbert space, let C be a closed convex subset of H and let P_C be the metric projection of H onto C. Let γ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be an ($\frac{3}{2}$, $\frac{1}{2}$, γ)-super hybrid mapping, i.e.,*

$$3\|Sx - Sy\|^2 + 4\gamma\langle x - y, Sx - Sy\rangle \leq \|x - Sy\|^2 + \|Sx - y\|^2 + (1 + 4\gamma)\|x - y\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\liminf_{n\to\infty} \alpha_n(1 - \alpha_n) > 0$. *Suppose $\{x_n\}$ is the sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = P_C\big(\alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n)\big), \quad n = 1, 2, \dots.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(S)$, where $v = \lim_{n\to\infty} P_{F(S)}x_n$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Next, we prove a weak convergence theorem of Mann's type for a class of mappings containing the class of nonexpansive mappings in a Hilbert space. Before proving it, we state the following lemma [20].

Lemma 4.5. *Let $\{\alpha_n\} \subset [0, \infty)$ and $\{\beta_n\} \subset [0, \infty)$ be sequences of real numbers such that $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\sum_{n=1}^{\infty} \alpha_n\beta_n < \infty$. Then* $\liminf_{n\to\infty} \alpha_n = 0$.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let γ be a real number with $\gamma \neq -1$ and let $S : C \to H$ be a mapping such that*

$$\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2$$

for all $x, y \in C$. Let $\{\alpha_n\}$ be a sequence of real numbers such that $0 \leq \alpha_n \leq 1$ and $\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n) = \infty$. Suppose $\{x_n\}$ is a sequence generated by $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)P_C\big(\frac{1}{1+\gamma}Sx_n + \frac{\gamma}{1+\gamma}x_n\big), \quad n = 1, 2, \dots.$$

If $F(S) \neq \emptyset$, then the sequence $\{x_n\}$ converges weakly to an element v of $F(S)$, where $v = \lim_{n\to\infty} P_{F(S)}x_n$ and $P_{F(S)}$ is the metric projection of H onto $F(S)$.

Proof. We have that for any $x, y \in C$,

$$\begin{aligned}
&\|Sx - Sy\|^2 + 2\gamma\langle x - y, Sx - Sy\rangle \leq (1 + 2\gamma)\|x - y\|^2 \\
&\qquad \Longleftrightarrow \|Sx - Sy\|^2 + \gamma(\|x - Sy\|^2 + \|Sx - y\|^2 - \|Sx - x\|^2 - \|y - Sy\|^2) \\
&\qquad\qquad \leq (1 + 2\gamma)\|x - y\|^2 \\
&\qquad \Longleftrightarrow \|Sx - Sy\|^2 + \gamma\|x - Sy\|^2 \\
&\qquad\qquad \leq -\gamma\|Sx - y\|^2 + (1 + 2\gamma)\|x - y\|^2 + \gamma\|Sx - x\|^2 + \gamma\|y - Sy\|^2.
\end{aligned}$$

So, S is a (1, 0, γ)-super hybrid mapping of C into H. Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have from Theorem 3.1 that T is a (1, 0)-generalized hybrid mapping of C into H, i.e., T is a nonexpansive mapping of C into H. Further, we have $F(S) = F(T)$. Let $z \in F(T)$. Since T is quasi-nonexpansive, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)P_C T x_n - z\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|P_C T x_n - z\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 \\
&= \|x_n - z\|^2
\end{aligned}$$

for all $n \in \mathbb{N}$. Hence, $\lim_{n\to\infty} \|x_n - z\|^2$ exists. So, we have that $\{x_n\}$ is bounded. We also have from (2.1) that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \|\alpha_n x_n + (1 - \alpha_n)P_C T x_n - z\|^2 \\
&= \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|P_C T x_n - z\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2 \\
&\leq \alpha_n\|x_n - z\|^2 + (1 - \alpha_n)\|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2 \\
&= \|x_n - z\|^2 - \alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2.
\end{aligned}$$

So, we have

$$\alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2 \leq \|x_n - z\|^2 - \|x_{n+1} - z\|^2.$$

Summing up these inequalities with respect to $n = 1, 2, \ldots, N$, we have

$$\sum_{n=1}^{N} \alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2 \leq \|x_1 - z\|^2 - \|x_{N+1} - z\|^2.$$

Putting $c = \lim_{n\to\infty} \|x_n - z\|^2$ and letting $N \to \infty$, we obtain

$$\sum_{n=1}^{\infty} \alpha_n(1 - \alpha_n)\|P_C T x_n - x_n\|^2 \leq \|x_1 - z\|^2 - c < \infty.$$

From the assumptions of $\{\alpha_n\}$ and Lemma 4.5, we have

$$\liminf_{n\to\infty} \|P_C T x_n - x_n\|^2 = 0.$$

On the other hand, we have from $x_{n+1} - x_n = (1-\alpha_n)\|P_C T x_n - x_n\|$ that

$$\begin{aligned}
&\|P_C T x_{n+1} - x_{n+1}\| \\
&\qquad = \alpha_n \|P_C T x_{n+1} - x_n\| + (1-\alpha_n)\|P_C T x_{n+1} - P_C T x_n\| \\
&\qquad \le \alpha_n (\|P_C T x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\|) \\
&\qquad\quad + (1-\alpha_n)\|P_C T x_{n+1} - P_C T x_n\| \\
&\qquad \le \alpha_n \|P_C T x_{n+1} - x_{n+1}\| + \alpha_n \|x_{n+1} - x_n\| + (1-\alpha_n)\|x_{n+1} - x_n\| \\
&\qquad = \alpha_n \|P_C T x_{n+1} - x_{n+1}\| + \|x_{n+1} - x_n\| \\
&\qquad = \alpha_n \|P_C T x_{n+1} - x_{n+1}\| + (1-\alpha_n)\|P_C T x_n - x_n\|.
\end{aligned}$$

So, we have $(1-\alpha_n)\|P_C T x_{n+1} - x_{n+1}\| \le (1-\alpha_n)\|P_C T x_n - x_n\|$. Then, we have $\|P_C T x_{n+1} - x_{n+1}\| \le \|P_C T x_n - x_n\|$. So, $\lim_{n\to\infty} \|P_C T x_n - x_n\|^2$ exists. Then, we obtain that

$$\lim_{n\to\infty} \|P_C T x_n - x_n\|^2 = \liminf_{n\to\infty} \|P_C T x_n - x_n\|^2 = 0.$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v$ for some $v \in C$. Since $\lim_{n\to\infty} \|P_C T x_n - x_n\| = 0$ and $P_C T$ is nonexpansive, we have from Theorem 4.1 that v is a fixed point of $P_C T$, i.e., $P_C T v = v$. We have from (2.5) that for $u \in F(T)$,

$$\begin{aligned}
2\|v-u\|^2 &= 2\|P_C T v - u\|^2 \\
&\le 2\langle Tv - u, P_C T v - u\rangle \\
&= \|Tv - u\|^2 + \|P_C T v - u\|^2 - \|Tv - P_C T v\|^2
\end{aligned}$$

and hence

$$2\|v-u\|^2 \le \|v-u\|^2 + \|v-u\|^2 - \|Tv - v\|^2.$$

So, we have $0 \le -\|Tv - v\|^2$. and hence $Tv = v$.

Let $\{x_{n_i}\}$ and $\{x_{n_j}\}$ be two subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup v_1$ and $x_{n_j} \rightharpoonup v_2$. To complete the proof, we show $v_1 = v_2$. We know $v_1, v_2 \in F(T)$ and hence $\lim_{n\to\infty} \|x_n - v_1\|$ and $\lim_{n\to\infty} \|x_n - v_2\|$ exist. Assume $v_1 \ne v_2$. Then, we have from Theorem 2.1 that

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - v_1\| &= \lim_{i\to\infty} \|x_{n_i} - v_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - v_2\| \\
&= \lim_{n\to\infty} \|x_n - v_2\| \\
&= \lim_{j\to\infty} \|x_{n_j} - v_2\| \\
&< \lim_{j\to\infty} \|x_{n_j} - v_1\| \\
&= \lim_{n\to\infty} \|x_n - v_1\|.
\end{aligned}$$

This is a contradiction. So, we obtain $v_2 = v_1$. This implies that $\{x_n\}$ converges weakly to an element v of $F(T)$. Since $\|x_{n+1} - z\| \le \|x_n - z\|$ for all $z \in F(T)$ and $n \in \mathbb{N}$, we obtain from Takahashi and Toyoda [24] that $\{P_{F(T)} x_n\}$ converges strongly to an element p of $F(T)$. On the other hand, we have from the property of $P_{F(T)}$ that

$$\langle x_n - P_{F(T)}x_n, P_{F(T)}x_n - u\rangle \geq 0$$

for all $u \in F(T)$ and $n \in \mathbb{N}$. Since $x_n \rightharpoonup v$ and $P_{F(T)}x_n \to p$, we obtain

$$\langle v - p, p - u\rangle \geq 0$$

for all $u \in F(T)$. Putting $u = v$, we obtain $p = v$. This means $v = \lim_{n\to\infty} P_{F(T)}x_n$. This completes the proof. □

5. Strong convergence theorems

In this section, using the hybrid method by Nakajo and Takahashi [16], we first prove a strong convergence theorem for super hybrid mappings with an equilibrium problem in a Hilbert space.

Theorem 5.1. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let α, β and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $EP(f) \cap F(S) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} f(z_n, y) + \frac{1}{\lambda_n}\langle z_n - x_n, y - z_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1-\alpha_n)(\frac{1}{1+\gamma}Sz_n + \frac{\gamma}{1+\gamma}z_n), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n\cap Q_n}x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0,\infty)$ satisfy

$$0 \leq \alpha_n \leq a < 1 \quad \textit{and} \quad 0 < b \leq \lambda_n$$

for some $a, b \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(f)}x$, where $P_{F(S)\cap EP(f)}$ is the metric projection of H onto $F(S) \cap EP(f)$.

Proof. Put $T = \frac{1}{1+\gamma}S + \frac{\gamma}{1+\gamma}I$. Then, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into H and $F(S) = F(T)$. Since $F(T)$ is closed and convex, $F(S) \cap EP(f)$ is closed and convex. So, there exists the mertic projection oh H onto $F(S) \cap EP(f)$. Further, we have

$$y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$$

for all $n \in \mathbb{N}$. From

$$\|y_n - z\|^2 \leq \|x_n - z\|^2$$
$$\Longleftrightarrow \|y_n\|^2 - \|x_n\|^2 - 2\langle y_n - x_n, z\rangle \leq 0,$$

we have that C_n, Q_n and $C_n \cap Q_n$ are closed and convex for all $n \in \mathbb{N}$. We next show that $C_n \cap Q_n$ is nonempty. Let $z \in F(T) \cap EP(f)$. Put $z_n = T_{\lambda_n}x_n$ for each $n \in \mathbb{N}$. From $z = T_{\lambda_n}z$ and Lemma 2.2, we have that for any $n \in \mathbb{N}$,

$$\|z_n - z\|^2 = \|T_{\lambda_n}x_n - z\|^2 \leq \|x_n - z\|^2. \tag{5.1}$$

Since T is quasi-nonexpansive, we have from (5.1) that

$$\begin{aligned}\|y_n - z\|^2 &= \|\alpha_n x_n + (1-\alpha_n)Tz_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2 \\ &\le \alpha_n\|x_n - z\|^2 + (1-\alpha_n)\|x_n - z\|^2 \\ &= \|x_n - z\|^2.\end{aligned}$$

So, we have $z \in C_n$ and hence $F(T) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show by induction that $F(T) \cap EP(f) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. From $F(T) \cap EP(f) \subset Q_1$, it follows that $F(T) \cap EP(f) \subset C_1 \cap Q_1$. Suppose that $F(T) \cap EP(f) \subset C_k \cap Q_k$ for some k. From $x_{k+1} = P_{C_k \cap Q_k} x$, we have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in C_k \cap Q_k.$$

Since $F(T) \cap EP(f) \subset C_k \cap Q_k$, we also have

$$\langle x_{k+1} - z, x - x_{k+1}\rangle \ge 0, \quad \forall z \in F(T) \cap EP(f).$$

This implies $F(T) \cap EP(f) \subset Q_{k+1}$. So, we have $F(T) \cap EP(f) \subset C_{k+1} \cap Q_{k+1}$. By induction, we have $F(T) \cap EP(f) \subset C_n \cap Q_n$ for all $n \in \mathbb{N}$. This means that $\{x_n\}$ and $\{z_n\}$ are well-defined. Since $x_n = P_{Q_n} x$ and $x_{n+1} = P_{C_n \cap Q_n} x \subset Q_n$, we have from (2.2) that

$$\begin{aligned}0 &\le 2\langle x - x_n, x_n - x_{n+1}\rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2.\end{aligned}$$

So, we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{5.2}$$

Further, since $x_n = P_{Q_n} x$ and $z \in F(T) \cap EP(f) \subset Q_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{5.3}$$

So, we have that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$ are also bounded. From (2.5), we have

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|P_{Q_n} x - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - P_{Q_n} x\|^2 \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 \to 0.\end{aligned}$$

So, we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{5.4}$$

From $x_{n+1} \in C_n$, we have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. So, we get $\|y_n - x_{n+1}\| \to 0$. We also have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{5.5}$$

From $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|$ and $0 \le \alpha_n \le a < 1$, we have that

$$\|Tz_n - x_n\| \to 0. \tag{5.6}$$

Let $z \in F(T) \cap EP(f)$. Using $z_n = T_{\lambda_n} x_n$ and Lemma 2.4, we have that

$$\begin{aligned} \|x_n - z\|^2 &\geq \|x_n - T_{\lambda_n} x_n\|^2 + \|T_{\lambda_n} x_n - z\|^2 \\ &= \|x_n - z_n\|^2 + \|z_n - z\|^2 \end{aligned}$$

and hence

$$\|x_n - z_n\|^2 \leq \|x_n - z\|^2 - \|z_n - z\|^2.$$

From $\|y_n - z\|^2 \leq \alpha_n \|x_n - z\|^2 + (1 - \alpha_n)\|z_n - z\|^2$ and hence

$$\|z_n - z\|^2 \geq \frac{\|y_n - z\|^2 - \alpha_n \|x_n - z\|^2}{1 - \alpha_n},$$

we have

$$\begin{aligned} \|x_n - z_n\|^2 &\leq \|x_n - z\|^2 - \frac{\|y_n - z\|^2 - \alpha_n \|x_n - z\|^2}{1 - \alpha_n} \\ &= \frac{\|x_n - z\|^2 - \|y_n - z\|^2}{1 - \alpha_n}. \end{aligned}$$

We also have

$$\begin{aligned} \|x_n - z\|^2 - \|y_n - z\|^2 &= \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 - \|y_n\|^2 + 2\langle y_n, z\rangle - \|z\|^2 \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, z\rangle \\ &\leq |\|x_n\|^2 - \|y_n\|^2| + 2|\langle x_n - y_n, z\rangle| \\ &\leq \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|x_n - y_n\|\|z\|. \end{aligned}$$

Since $\|x_n - z\|^2 - \|y_n - z\|^2 \geq 0$ and $\lim_{n\to\infty} \|x_n - y_n\| = 0$, we have

$$\lim_{n\to\infty} (\|x_n - z\|^2 - \|y_n - z)\|^2 = 0. \tag{5.7}$$

Since $0 \leq \alpha_n \leq a < 1$, from (5.7) we have $\lim_{n\to\infty} \|x_n - z_n\|^2 = 0$. So, we have

$$\|x_n - z_n\| \to 0. \tag{5.8}$$

Since $y_n = \alpha_n x_n + (1 - \alpha_n)Tz_n$, we have $y_n - Tz_n = \alpha_n(x_n - Tz_n)$. So, from (5.6) we have

$$\|y_n - Tz_n\| = \alpha_n \|x_n - Tz_n\| \to 0. \tag{5.9}$$

Since

$$\|z_n - Tz_n\| \leq \|z_n - x_n\| + \|x_n - y_n\| + \|y_n - Tz_n\|,$$

from (5.5), (5.8) and (5.9) we have

$$\|z_n - Tz_n\| \to 0. \tag{5.10}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such that $x_{n_i} \rightharpoonup z^*$. We have from (5.8) and $x_{n_i} \rightharpoonup z^*$ that $z_{n_i} \rightharpoonup z^*$. From (5.10), we have $z^* \in F(T)$. Next, let us show $z^* \in EP(f)$. Since $z_n = T_{\lambda_n} x_n$, we have that for any $y \in C$,

$$f(z_n, y) + \frac{1}{\lambda_n} \langle y - z_n, z_n - x_n\rangle \geq 0.$$

From (A2), we have

$$\frac{1}{\lambda_n} \langle y - z_n, z_n - x_n\rangle \geq f(y, z_n).$$

From $0 < b \leq \lambda_n$ and (5.8), we have

$$\lim_{n\to\infty} \frac{z_n - x_n}{\lambda_n} = 0.$$

So, from (A4) we have

$$0 \geq f(y, z^*). \tag{5.11}$$

Put $z_t^* = ty + (1-t)z^*$ for all $t \in (0,1]$ and $y \in C$. Since C is convex, we have $z_t^* \in C$. From (A1), (A4) and (5.11), we have

$$\begin{aligned} 0 = f(z_t^*, z_t^*) &\leq tf(z_t^*, y) + (1-t)f(z_t^*, z^*) \\ &\leq tf(z_t^*, y) \end{aligned}$$

and hence

$$0 \leq f(z_t^*, y).$$

Letting $t \to 0$, from (A3) we have that for each $y \in C$,

$$0 \leq f(z^*, y). \tag{5.12}$$

This implies $z^* \in EP(f)$. So, we have $z^* \in F(T) \cap EP(f)$. Put $z_0 = P_{F(T)\cap EP(f)}x$. Since $z_0 = P_{F(T)\cap EP(f)}x \subset C_n \cap Q_n$ and $x_{n+1} = P_{C_n \cap Q_n}x$, we have that

$$\|x - x_{n+1}\|^2 \leq \|x - z_0\|^2. \tag{5.13}$$

Since $\|\cdot\|^2$ is weakly lower semicontinuous, from $x_{n_i} \rightharpoonup z^*$ we have that

$$\begin{aligned} \|x - z^*\|^2 &= \|x\|^2 - 2\langle x, z^*\rangle + \|z^*\|^2 \\ &\leq \liminf_{i\to\infty}(\|x\|^2 - 2\langle x, x_{n_i}\rangle + \|x_{n_i}\|^2) \\ &= \liminf_{i\to\infty} \|x - x_{n_i}\|^2 \\ &\leq \|x - z_0\|^2. \end{aligned}$$

From the definition of z_0, we obtain $z^* = z_0$. So, we obtain $x_n \rightharpoonup z_0$. We finally show that $x_n \to z_0$. We have

$$\|z_0 - x_n\|^2 = \|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle, \quad \forall n \in \mathbb{N}.$$

Since $x_n = P_{Q_n}x$ and $z_0 \in F(T) \cap EP(f) \subset Q_n$, we have

$$\|x - x_n\|^2 \leq \|x - z_0\|^2 \tag{5.14}$$

and hence

$$\begin{aligned} \limsup_{n\to\infty} \|z_0 - x_n\|^2 &= \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - x_n\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &\leq \limsup_{n\to\infty}(\|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - x_n\rangle) \\ &= \|z_0 - x\|^2 + \|x - z_0\|^2 + 2\langle z_0 - x, x - z_0\rangle \\ &= \|z_0 - z_0\|^2 = 0. \end{aligned}$$

So, we obtain $\lim_{n\to\infty} \|z_0 - x_n\| = 0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

Next, we prove a strong convergence theorem by the shrinking projection method [23].

Theorem 5.2. *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1), (A2), (A3) and (A4). Let α, β and γ be real numbers with $\gamma \neq -1$ and let $S : C \to H$ be an (α, β, γ)-super hybrid mapping such that $EP(f) \cap F(S) \neq \emptyset$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 = x \in C$ and*

$$\begin{cases} f(z_n, y) + \frac{1}{\lambda_n} \langle z_n - x_n, y - z_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)(\frac{1}{1+\gamma} S z_n + \frac{\gamma}{1+\gamma} z_n), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1}, and $\{\alpha_n\} \subset [0,1]$ and $\{\lambda_n\} \subset [0, \infty)$ are sequences such that

$$\liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < b \leq \lambda_n$$

for some $a, b \in \mathbb{R}$. Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap EP(f)} x$, where $P_{F(S)\cap EP(f)}$ is the metric projection of H onto $F(S) \cap EP(f)$.

Proof. Put $T = \frac{1}{1+\gamma} S + \frac{\gamma}{1+\gamma} I$. Then, we have from Theorem 3.1 that T is an (α, β)-generalized hybrid mapping of C into H and $F(S) = F(T)$. Since $F(T)$ is closed and convex, so is $F(S)$. Then, $F(S) \cap EP(f)$ is closed and convex. So, there exists the mertic projection of H onto $F(S) \cap EP(f)$. Further, we have

$$y_n = \alpha_n x_n + (1 - \alpha_n) T z_n$$

for all $n \in \mathbb{N}$. Put $z_n = T_{\lambda_n} x_n$ for each $n \in \mathbb{N}$ and take $z \in F(T) \cap EP(f)$. From $z = T_{\lambda_n} z$ and Lemma 2.2, we have that for any $n \in \mathbb{N}$,

$$\|z_n - z\| = \|T_{\lambda_n} x_n - z\| \leq \|x_n - z\|. \tag{5.15}$$

We shall show that C_n are closed and convex, and $F(T) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N}$. It is obvious from the assumption that $C_1 = C$ is closed and convex, and $F(T) \cap EP(f) \subset C_1$. Suppose that C_k is closed and convex, and $F(T) \cap EP(f) \subset C_k$. From Nakajo and Takahashi [16], we know that for $z \in C_k$,

$$\begin{aligned} &\|y_k - z\|^2 \leq \|x_k - z\|^2 \\ \Longleftrightarrow &\|y_k\|^2 - \|x_k\|^2 - 2\langle y_k - x_k, z\rangle \leq 0. \end{aligned}$$

So, C_{k+1} is closed and convex. If $z \in F(T) \cap EP(f) \subset C_k$, then we have from (5.15) that

$$\begin{aligned} \|y_n - z\| &= \|\alpha_n x_n + (1 - \alpha_n) T z_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) \|z_n - z\| \\ &\leq \alpha_n \|x_n - z\| + (1 - \alpha_n) \|x_n - z\| \\ &= \|x_n - z\|. \end{aligned}$$

Hence, we have $z \in C_{k+1}$. By induction, we have that C_n are closed and convex, and $F(T) \cap EP(f) \subset C_n$ for all $n \in \mathbb{N}$. Since C_n is closed and convex, there exists the metric projection P_{C_n} of H onto C_n. Thus, $\{x_n\}$ is well-defined.

Since $\{C_n\}$ is a nonincreasing sequence of nonempty closed convex subsets of H with respect to inclusion, it follows that

$$\emptyset \neq F(T) \cap EP(f) \subset \text{M-}\lim_{n\to\infty} C_n = \bigcap_{n=1}^{\infty} C_n. \tag{5.16}$$

Put $C_0 = \bigcap_{n=1}^{\infty} C_n$. Then, by Theorem 2.3 we have that $\{P_{C_n}x\}$ converges strongly to $x_0 = P_{C_0}x$, i.e.,

$$x_n = P_{C_n}x \to x_0.$$

To complete the proof, it is sufficient to show that $x_0 = P_{F(T)\cap EP(f)}x$. Since $x_n = P_{C_n}x$ and $x_{n+1} = P_{C_{n+1}}x \in C_{n+1} \subset C_n$, we have (2.2) that

$$\begin{aligned} 0 &\le 2\langle x - x_n, x_n - x_{n+1}\rangle \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 - \|x_n - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - x_n\|^2. \end{aligned}$$

So, we get that

$$\|x - x_n\|^2 \le \|x - x_{n+1}\|^2. \tag{5.17}$$

Further, since $x_n = P_{C_n}x$ and $z \in F(T) \cap EP(f) \subset C_n$, we have

$$\|x - x_n\|^2 \le \|x - z\|^2. \tag{5.18}$$

So, we have that $\lim_{n\to\infty} \|x - x_n\|^2$ exists. This implies that $\{x_n\}$ is bounded. Hence, $\{y_n\}$, $\{z_n\}$ and $\{Tz_n\}$ are also bounded. From Lemma 2.5, we have

$$\begin{aligned} \|x_n - x_{n+1}\|^2 &= \|P_{C_n}x - x_{n+1}\|^2 \\ &\le \|x - x_{n+1}\|^2 - \|x - P_{C_n}x\|^2 \\ &= \|x - x_{n+1}\|^2 - \|x - x_n\|^2 \to 0. \end{aligned}$$

So, we have that

$$\|x_n - x_{n+1}\|^2 \to 0. \tag{5.19}$$

From $x_{n+1} \in C_{n+1}$, we also have that $\|y_n - x_{n+1}\| \le \|x_n - x_{n+1}\|$. So, we get that $\|y_n - x_{n+1}\| \to 0$. Using this, we have

$$\|y_n - x_n\| \le \|y_n - x_{n+1}\| + \|x_{n+1} - x_n\| \to 0. \tag{5.20}$$

From $\liminf_{n\to\infty} \alpha_n < 1$, there exist a subsequence $\{\alpha_{n_i}\}$ of $\{\alpha_n\}$ and α_0 with $0 \le \alpha_0 < 1$ such that $\alpha_{n_i} \to \alpha_0$. Since $\|x_n - y_n\| = \|x_n - \alpha_n x_n - (1-\alpha_n)Tz_n\| = (1-\alpha_n)\|x_n - Tz_n\|$, we also have that

$$\|Tz_{n_i} - x_{n_i}\| \to 0. \tag{5.21}$$

Let $z \in F(T) \cap EP(f)$. Using $z_n = T_{\lambda_n}x_n$ and Lemma 2.4, we have that

$$\begin{aligned} \|x_n - z\|^2 &\ge \|x_n - T_{\lambda_n}x_n\|^2 + \|T_{\lambda_n}x_n - z\|^2 \\ &= \|x_n - z_n\|^2 + \|z_n - z\|^2 \end{aligned}$$

and hence

$$\|x_n - z_n\|^2 \le \|x_n - z\|^2 - \|z_n - z\|^2.$$

We also have $\|y_n - z\|^2 \le \alpha_n \|x_n - z\|^2 + (1-\alpha_n)\|z_n - z\|^2$ and hence

$$\|z_{n_i} - z\|^2 \ge \frac{\|y_{n_i} - z\|^2 - \alpha_{n_i}\|x_{n_i} - z\|^2}{1-\alpha_{n_i}}.$$

Therefore, we have

$$\begin{aligned}\|x_{n_i} - z_{n_i}\|^2 &\le \|x_{n_i} - z\|^2 - \frac{\|y_{n_i} - z\|^2 - \alpha_{n_i}\|x_{n_i} - z\|^2}{1-\alpha_{n_i}} \\ &= \frac{\|x_{n_i} - z\|^2 - \|y_{n_i} - z\|^2}{1-\alpha_{n_i}}.\end{aligned}$$

We also have

$$\begin{aligned}\|x_n - z\|^2 - \|y_n - z\|^2 &= \|x_n\|^2 - 2\langle x_n, z\rangle + \|z\|^2 - \|y_n\|^2 + 2\langle y_n, z\rangle - \|z\|^2 \\ &= \|x_n\|^2 - \|y_n\|^2 - 2\langle x_n - y_n, z\rangle \\ &\le |\|x_n\|^2 - \|y_n\|^2| + 2|\langle x_n - y_n, z\rangle| \\ &\le \|x_n - y_n\|(\|x_n\| + \|y_n\|) + 2\|x_n - y_n\|\|z\|.\end{aligned}$$

Since $0 \le \|x_n - z\|^2 - \|y_n - z\|^2$, from (5.20) we have

$$\lim_{n\to\infty} (\|x_n - z\|^2 - \|y_n - z)\|^2 = 0. \tag{5.22}$$

Since $\alpha_{n_i} \to \alpha_0$ and $0 \le \alpha_0 < 1$, we have

$$\|x_{n_i} - z_{n_i}\| \to 0. \tag{5.23}$$

From $y_n = \alpha_n x_n + (1-\alpha_n)Tz_n$, we have $y_n - Tz_n = \alpha_n(x_n - Tz_n)$. So, from (5.21) we have

$$\|y_{n_i} - Tz_{n_i}\| = \alpha_{n_i}\|x_{n_i} - Tz_{n_i}\| \to 0. \tag{5.24}$$

Since

$$\|z_{n_i} - Tz_{n_i}\| \le \|z_{n_i} - x_{n_i}\| + \|x_{n_i} - y_{n_i}\| + \|y_{n_i} - Tz_{n_i}\|,$$

from (5.20), (5.23) and (5.24) we have

$$\|z_{n_i} - Tz_{n_i}\| \to 0. \tag{5.25}$$

Since $x_{n_i} = P_{C_{n_i}} x \to x_0$, we have $z_{n_i} \to x_0$. So, from (5.25) and Lemma 4.1 we have $x_0 \in F(T)$. Next, let us show $x_0 \in EP(f)$. We know $z_{n_i} \to x_0$. We have from $z_n = T_{\lambda_n} x_n$ that for any $y \in C$,

$$f(z_n, y) + \frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge 0.$$

From (A2), we have

$$\frac{1}{\lambda_n}\langle y - z_n, z_n - x_n\rangle \ge f(y, z_n).$$

From $0 < b \le \lambda_n$ and (5.23), we know

$$\lim_{n\to\infty} \frac{z_{n_i} - x_{n_i}}{\lambda_{n_i}} = 0.$$

So, we have

$$0 \ge f(y, x_0). \tag{5.26}$$

Put $z_t = ty + (1-t)x_0$ for all $t \in (0,1]$ and $y \in C$. Since C is convex, we have $z_t \in C$. From (A1), (A4) and (5.26), we have

$$\begin{aligned} 0 = f(z_t, z_t) &\le t f(z_t, y) + (1-t) f(z_t, x_0) \\ &\le t f(z_t, y) \end{aligned}$$

and hence

$$0 \le f(z_t, y).$$

Letting $t \to 0$, we have from (A3) that for each $y \in C$,

$$0 \le f(x_0, y). \tag{5.27}$$

This implies $x_0 \in EP(f)$. So, we have that $x_0 \in F(T) \cap EP(f)$. Put $z_0 = P_{F(T)\cap EP(f)}x$. Since $z_0 = P_{F(T)\cap EP(f)}x \subset C_{n+1}$ and $x_{n+1} = P_{C_{n+1}}x$, we have that

$$\|x - x_{n+1}\|^2 \le \|x - z_0\|^2. \tag{5.28}$$

So, we have that

$$\|x - x_0\|^2 = \lim_{n\to\infty} \|x - x_n\|^2 \le \|x - z_0\|^2.$$

So, we get $z_0 = x_0$. Hence, $\{x_n\}$ converges strongly to z_0. This completes the proof. □

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